ON THE SOLUTION OF CERTAIN CONTACT PROBLEMS OF THE THEORY OF ELASTICITY

(K RESHENIIU NEKOTORYKH KONTAKTNYKH ZADACH TEORII UPRUGOSTI)

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The following contact problems of the theory of elasticity are considered: (1) the problem of the effect of a stamp on an elastic strip of thickness h lying frictionless on a rigid base; (2) the problem of the effect of a stamp on an elastic strip of thickness h attached rigidly to a rigid base; (3) the axisymmetric problem of the interaction of a belt with an elastic cylinder of radius R.

These problems can be reduced, by operational calculus methods, to the solution of the following integral equation:

$$\int_{-a}^{a} q(\xi) K\left(\frac{x-\xi}{l}\right) d\xi = \pi\Delta\delta(x), \qquad |x| \leqslant a, \quad \Delta = \frac{E}{2(1-\sigma^2)} \tag{1}$$

Here a is half the length of the line of contact; $q(\xi)$ the contact pressure; $\delta(x)$ the setting of the strip or cylinder surface in the contact domain; $l = h$ or $l = R$. The kernel $K(k)$ of equation (1) has the form

$$K(k) = \int_0^\infty \frac{L(u)}{u} \cos ku\, du, \qquad \text{where } L(u) \to 1 \quad \text{as} \quad u \to \infty \tag{2}$$

For values of the parameter $\lambda = l/a > 1$, there are a number of methods which can be used to obtain approximate solutions of equations (1) of problems (1) to (3) with sufficient simplicity. However, for $\lambda < 1$, one of these methods becomes not at all applicable, and the others require more and more awkward transformations and calculations to obtain solutions of the required accuracy as λ diminishes. The only relief here is that for very small values of λ a very simple degenerate expression of the form

$$q(x) = \frac{\Delta\delta(x)}{Al}, \qquad A = \lim_{u\to 0}\frac{L(u)}{u} \tag{3}$$

can be found for all the above-mentioned problems.

As will be established below, this solution yields practically exact results for $\lambda \leqslant 1.5$. It should just be noted that the degenerate solution (3) has an essential disadvantage. If $\delta'(x)$ satisfies the Holder condition, then for any value of $\lambda \in (0, \infty)$ the solution of (1) for problems (1) to (3) should have a singularity of the form $(a^2 - x^2)^{-1/2}$, as can be shown. Hence it follows that the solution (3) for values of x close to $k \pm a$ will yield erroneous results.

Starting from the above, let us set ourselves the task of obtaining a practical, convenient and approximate solution of equation (1) suitable for the values $0 < \lambda \leqslant 1$, and having a singularity of the form $(a^2 - x^2)^{-1/2}$.

Keeping in mind the results of Krein [1], let us try to obtain the mentioned solution of (1) for the case $\delta(x) = \delta = \text{const}$. This solution should evidently have the form

$$q(x) = \frac{\Delta\delta}{Al}\,[1 + f_+(x) + f_-(x)] \tag{4}$$

where the functions f_+ and f_- have the following properties:

the function $f_+(x)$ has a singularity of the form $(a - x)^{-1/2}$ and tends rapidly to zero for $x < a$;

the function $f_-(x)$ has a singularity of the form $(a + x)^{-1/2}$ and tends rapidly to zero for $x > -a$.

Thus the problem has been reduced to finding the functions f_+ and f_- with the mentioned properties.

Let us represent (1) in other variables

$$\int_0^{2/\lambda} q(l\tau - a)\,K(t-\tau)\,d\tau = \frac{\pi\Delta\delta}{l} \qquad \left(t = \frac{a+x}{l},\ \tau = \frac{a+\xi}{l}\right) \tag{5}$$

Let us rewrite (5) as

$$\int_0^{\infty} q(l\tau - a)\,K(t-\tau)\,d\tau = \frac{\pi\Delta\delta}{l} + \int_{2/\lambda}^{\infty} q(l\tau - a)\,K(t-\tau)\,d\tau \tag{6}$$

Solving the integral equation (6) for small values of λ by successive approximations, it is possible to be limited, for subsequent accuracy, to the zero approximation which is determined from the equation

$$\int_0^\infty q\,(l\tau - a)\,K\,(t-\tau)\,d\tau = \frac{\pi\Delta\delta}{l} \tag{7}$$

The integral equation (7) is the equation of the problem of the effect of a semi-infinite stamp $(-a \leqslant x < \infty)$ on an elastic strip, or the problem of the interaction of a semi-infinite belt $(-a \leqslant x < \infty)$ with an elastic cylinder. This indicates that for small λ the influence of the right end of the stamp or belt on the state of stress under the left end (and conversely) can be neglected.

Therefore, the contact pressure distribution $q(x)$ under the left end is described well enough by the solution of (7), which evidently has the form

$$q\,(x) = \frac{\Delta\delta}{Al}\,[1 + f_-(x)] \tag{8}$$

The solution of (7) can be obtained in closed form by the Wiener-Hopf method and, therefore, the function $f_-(x)$ can be determined. The function $f_+(x)$ can also be found in an analogous manner. Hence, a solution of the form (4) of equations (1), which is valid for small values of λ, can always be constructed.

In order to obtain a solution suitable for practical calculations, let us proceed as follows. Let us introduce the function [2]

$$\Lambda\,(u) = u^{-1}L\,(u)\,\sqrt{u^2 + A^{-2}} \tag{9}$$

This function has the following properties:

$$\Lambda\,(u) \to 1 \quad \text{as} \quad u \to \infty \text{ and } u \to 0$$

Moreover, it can easily be shown by direct calculations that the function $\Lambda(u)$ deviates by not more than 20 per cent from 1 for all values of $u \in (0, \infty)$ for the problems (1), (2) and (3) listed above, so that it is possible to put $\Lambda(u) \equiv 1$ with sufficient accuracy for the sequel (the error in the final results does not exceed 6 per cent for the most unfavorable cases). Then the kernel of (7) will be

$$K\,(k) = \int_0^\infty \frac{\cos ku\,du}{\sqrt{u^2 + A^{-2}}} \tag{10}$$

and its solution is given by the formula ($\Phi(x)$ is the probability integral)

$$q\,(x) = \frac{\Delta\delta}{Al}\left[\Phi\left(\sqrt{\frac{2\,(a+x)}{Al}}\right)^{1/2} + \frac{1}{\sqrt{\pi}}\left(\frac{a+x}{Al}\right)^{-1/2}\exp\frac{-(a+x)}{Al}\right] \tag{11}$$

Now, the solution of (1) of type (4) can easily be obtained without difficulty and we represent it in a form convenient for calculations

$$q(x) = \frac{\Delta\delta}{\sqrt{a^2 - x^2}}\,\omega\left(\frac{x}{a}\right)$$

$$\omega\left(\frac{x}{a}\right) = [\Phi(\alpha\sqrt{2}) + \Phi(\beta\sqrt{2}) - 1]\,\alpha\beta + \frac{1}{\sqrt{\pi}}(\beta e^{-\alpha^2} + \alpha e^{-\beta^2})$$

$$\alpha = \sqrt{\frac{a+x}{Al}}, \qquad \beta = \sqrt{\frac{a-x}{Al}} \tag{12}$$

We obtain the quantity

$$P = \int_{-a}^{a} q(x)\,dx = \pi\Delta\delta\varkappa$$

$$\varkappa = \frac{D^2}{\pi}\left[(2 + D^{-2})\,\Phi(D) + \frac{2}{\sqrt{\pi}}D^{-1}e^{-D^2} - 1\right], \qquad D = \sqrt{\frac{2}{A\lambda}} \tag{13}$$

Calculations carried out showed convincingly that the solution obtained in (12) and (13) yields practically true results for $\lambda \leqslant 1$, where the accuracy rises as λ diminishes. The results of certain calculations are presented in Table 1. Given for comparison in the table are values of the quantity κ° calculated by methods from [3,4] (for problems (2) and (3) the calculations were carried out for $\sigma = 0.3$).

TABLE 1.

Problems	1)			2)	3)
λ	1	0.5	0.25	1	1
$\varkappa^{\circ}$	1.50	2.85	5.41	1.76	1.86
$\varkappa$	1.47	2.84	5.41	1.79	1.89

Now, on the basis of (13), let us determine the limits of applicability of the degenerate solution (3) which are mentioned in [3,4] with a certain margin. Starting from the relation

$$\frac{P - P^*}{P^*}\,100\% \leqslant 5\% \qquad (P^* = D^2\Delta\delta)$$

or

$$(2 + D)^{-2}\,\Phi(D) + \frac{2}{\sqrt{\pi}}D^{-1}e^{-D^2} \leqslant 2.05$$

TABLE 2.

Problems	1)	2)	3)
$\lambda_0{}^{\circ}$	$^1/_5$	$^1/_7$	$^1/_4$
λ_0	$^1/_5$	$^1/_4$	$^1/_4$

we find that the degenerate solution will yield practically accurate results for $\lambda \leqslant \lambda_0 = 0.1\,A^{-1}$. The results of calculating the quantity λ_0 for problems (1), (2) and (3) are given in Table 2 (it was assumed that $\sigma = 0.3$ for problems (2) and (3)); also presented in the table are values of $\lambda_0{}^{\circ}$ mentioned in [3,4].

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